Coriolis

The Coriolis term in the momentum equation is

$$\frac{du_i}{dt} = -2\,\varepsilon_{ijk}\,\Omega_j\,u_k \quad [+\text{ other terms}]$$

For the $i=1$ ($u$ component) case this becomes

$$\frac{du_1}{dt} = -2\,\varepsilon_{1jk}\,\Omega_j\,u_k \quad [+\text{ other}]$$

$\varepsilon_{1jk}$ is nonzero only for $(j,k) = (2,3)$ or $(j,k) = (3,2)$.
Then

$$\frac{du_1}{dt} = -2\,\varepsilon_{123}\,\Omega_2\,u_3 - 2\,\varepsilon_{132}\,\Omega_3\,u_2 \quad [+\text{ other}]$$

Recall $\Omega = [0, \omega\cos\phi, \omega\sin\phi]$. Then the $u$ component equation expands to

$$\frac{du_1}{dt} = -2\,\omega(\cos\phi)\,w + 2\,\omega(\sin\phi)\,v \quad [+\text{ other}]$$

For $i=2$ ($v$ component),

$$\frac{du_2}{dt} = -2\,\varepsilon_{2jk}\,\Omega_j\,u_k \quad [+\text{ other}]$$

$\varepsilon_{2jk}$ is nonzero only for $(j,k) = (1,3)$ or $(j,k) = (3,1)$.
From the definition of $\Omega$ we have $\Omega_j = 0$ for $j=1$.
Then the only nonzero case is $(j,k) = (3,1)$ and

$$\frac{du_2}{dt} = -2\,\varepsilon_{231}\,\Omega_3\,u_1 \,[+\text{ other}] = -2\,\omega\cos\phi\;v\,[+\text{other}]$$

For $i = 3$ ($w$ component),

$$\frac{du_3}{dt} = -2\,\varepsilon_{3jk}\,\Omega_j\,u_k \quad [+\text{ other}]$$

Here $\varepsilon_{3jk}$ is nonzero only for $(j,k) = (1,2)$ or $(j,k) = (2,1)$.
Then

$$\frac{du_3}{dt} = -2\,\varepsilon_{312}\,\Omega_1\,u_2 - 2\,\varepsilon_{321}\,\Omega_2\,u_1 \quad [+\text{ other}]$$

Again $\Omega_1 = 0$ so only the second term is non zero:

$$\frac{du_3}{dt} = -2\,\varepsilon_{321}\,\Omega_2\,u_1 \quad [+\text{ other}]$$

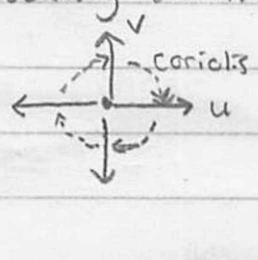
$$= 2\,\omega\,(\cos\phi)\,u \quad [+\text{other}]$$

Notice we have two pairs of Coriolis terms. In the $u$ and $v$ equations we have the terms that are proportional to the sine of the latitude ($\sin\phi$), and in the $u$ and $w$ equations we have terms proportional to $\cos\phi$.

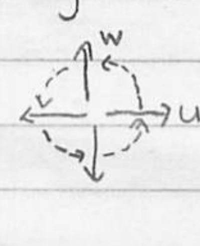
Each pair of terms causes momentum to be transferred from one component to another. but does not increase the total momentum of the atmosphere.

In both cases the velocity vector rotates by the Coriolis force but conserves its magnitude

looking down:




looking north



If we adopt the hydrostatic approximation then $-\frac{1}{\rho}\frac{\partial p}{\partial z} = -g$ and $\frac{dw}{dt} = 0$. This means the Coriolis term in the w equation must be neglected; otherwise, the term would cause $\frac{dw}{dt} \neq 0$.

Since the Coriolis terms work in pairs that conserve momentum, if we omit one term in the pair we must also omit the other term of the pair to preserve conservation of momentum. Thus if we adopt the hydrostatic approximation so that the $2\Omega(\cos\phi)u$ term is omitted, we must also omit $-2\omega(\cos\phi)w$ which appears in the u component equation.

To simplify notation we define $f_c \equiv 2\,\Omega \sin\phi$, which you will recognize as the usual form of the Coriolis parameter. Then our u and v equations are

$$\frac{du}{dt} = fv \quad [+ \text{ other}]$$

$$\frac{dv}{dt} = -fu \quad [+ \text{ other}]$$

How do we write this using summation notation? Notice the differing signs of the Coriolis terms in the u and v equations, suggesting that we can take advantage of the alternating unit tensor $\varepsilon_{ijk}$.

We want to ensure there is no Coriolis term for the w equation, i.e., $\varepsilon_{ijk} = 0$ for $i = 3$. We can do this by setting $k = 3$. Notice also that u is acted upon by v, and v is acted upon by u. Thus the index for the velocity variable in the equation for the $i$ component cannot be $i$ (that is, u does not act on itself, nor does v act on itself). Nor can the index be $k$, because we have set $k=3$. Then the index must be $j$.

Putting all this together yields

$$\frac{du_i}{dt} = f_c \, \varepsilon_{ij3} \, u_j \; [+ \text{ other}]$$

Check:

For $i = 1$, $$\frac{du_1}{dt} = f_c \, \varepsilon_{123} \, u_2 \; [+ \text{ other}] = f_c v \; [+ \text{ other}]$$

For $i = 2$, $$\frac{du_2}{dt} = f_c \, \varepsilon_{213} \, u_1 \; [+ \text{ other}] = -f_c u \; [+ \text{ other}]$$

Therefore Stull's statement "Often term IV is written as $+ f_c \, \varepsilon_{ij3} \, u_j$" (p. 78) really should be "If the hydrostatic approximation is adopted, term IV becomes $+ f_c \, \varepsilon_{ij3} \, u_j$"